Eigenfunctions Worked Example Differential Equations X. Du

- Suppose we have a one-dimensional box of length L, where $L \in \mathbb{R}^+$. Suppose we have the linear operator $D = \frac{d^2}{dx^2}$ subject to the boundary conditions $y(0) = 0$ and $y(L) = 0$. Find all eigenvalues and eigenfunctions y of D subject to these boundary conditions.
  - $Dy = \lambda y$ $\quad y'' = \lambda y$ $\quad y'' - \lambda y = 0$
  - Characteristic equation: $r^2 - \lambda = 0$. $\quad r = \pm\sqrt{\lambda}$
  - Case 1: $\lambda > 0$. $r = \pm\sqrt{\lambda}$. Two real roots. General solution: $y = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$. Substitute to solve BVP. $y(0) = c_1 + c_2 = 0$. $c_2 = -c_1$. $y(L) = c_1 e^{L\sqrt{\lambda}} + c_2 e^{-L\sqrt{\lambda}} = 0$. $c_1 e^{L\sqrt{\lambda}} - c_1 e^{-L\sqrt{\lambda}} = 0$. $c_1(e^{L\sqrt{\lambda}} - e^{-L\sqrt{\lambda}}) = 0$.
    - Case 1a: $c_1 = 0$. Then $c_2 = 0$, $y = 0$. This is the trivial solution.
    - Case 1b: $e^{L\sqrt{\lambda}} - e^{-L\sqrt{\lambda}} = 0$. $e^{L\sqrt{\lambda}} = e^{-L\sqrt{\lambda}}$. $L\sqrt{\lambda} = -L\sqrt{\lambda}$. $L\sqrt{\lambda} = 0$. Since L is positive, $L \neq 0$. $\lambda > 0 \Rightarrow \lambda \neq 0$ in this case, so $L\sqrt{\lambda} = 0$ is a contradiction. Case 1b does not exist.
  - Case 2: $\lambda = 0$. $r = 0$. One repeated root. General solution: $y = c_1 + c_2 x$ since $r = 0$. $y(0) = c_1 = 0$. $y(L) = c_2 L = 0$. Since $L \neq 0$, $c_2 = 0$. So, $y = 0$. Trivial solution again.
  - Case 3: $\lambda < 0$. $r = \pm i\sqrt{-\lambda}$. Two complex roots. General solution: $y = c_1 \cos\sqrt{-\lambda}x + c_2 \sin\sqrt{-\lambda}x$. Substitute. $y(0) = c_1 = 0$. $y(L) = c_2 \sin\sqrt{-\lambda}L = 0$.
    - Case 3a: $c_2 = 0$. Then $y = 0$. Trivial solution.
    - Case 3b: $\sin\sqrt{-\lambda}L = 0$. By the properties of the sine function, $\sqrt{-\lambda}L = n\pi$, where $n \in \mathbb{Z}^+$. $\sqrt{-\lambda} = \frac{n\pi}{L}$. $\lambda = -\frac{n^2\pi^2}{L^2}$. $y = c_2 \sin\frac{n\pi x}{L}$ is the general solution.
  - Answer: eigenvalues: $\lambda = -\frac{n^2\pi^2}{L^2}$, eigenfunctions: $y = c\sin\frac{n\pi x}{L}$
- Does this answer look familiar? It's a component of the Fourier sine series, and it's not a coincidence – this is one way of coming up with that specific component of the Fourier sine series. This has to do with properties of Hermitian operators and Sturm-Liouville theory.
  - However, note that this is not a Sturm-Liouville problem. Its boundary conditions do not meet the criteria.

- Try this example: Suppose we have a one-dimensional box of length $2L$, where $L \in \mathbb{R}^+$. Suppose we have the linear operator $D = \frac{d^2}{dx^2}$ subject to the periodic boundary conditions $y(-L) = y(L)$ and $y'(-L) = y'(L)$. Find all eigenvalues and eigenfunctions y of D subject to these boundary conditions.
  - Note that the answer should look familiar. Check your work if it doesn't.